4th Annual Lexington Mathematical Tournament
Individual Round

March 30th, 2013

1. What is the smallest positive integer divisible by 20, 12, and 13?
2. Two circles of radius 5 are placed in the plane such that their centers are 7 units apart. What is the largest possible distance between a point on one circle and a point on the other?
3. In a magic square, all the numbers in the rows, columns, and diagonals sum to the same value. How many 2×2 magic squares containing the integers $\{1, 2, 3, 4\}$ are there?
4. Ethan's sock drawer contains two pairs of white socks and one pair of red socks. Ethan picks two socks at random. What is the probability that he picks two white socks?
5. The sum of the time on a digital clock is the sum of the digits displayed on the screen. For example, the sum of the time at 10:23 would be 6. Assuming the clock is a 12 hour clock, what is the greatest possible positive difference between the sum of the time at some time and the sum of the time one minute later?
6. Given the expression $1 \div 2 \div 3 \div 4$, what is the largest possible resulting value if one were to place parentheses () somewhere in the expression?
7. At a convention, there are many astronomers, astrophysicists, and cosmologists. At first, all the astronomers and astrophysicists arrive. At this point, $\frac{3}{5}$ of the people in the room are astronomers. Then, all the cosmologists come, so now, 30% of the people in the room are astrophysicists. What fraction of the scientists are cosmologists?
8. At 10:00 AM, a minuteman starts walking down a 1200-step stationary escalator at 40 steps per minute. Halfway down, the escalator starts moving up at a constant speed, while the minuteman continues to walk in the same direction and at the same pace that he was going before. At 10:55 AM, the minuteman arrives back at the top. At what speed is the escalator going up, in steps per minute?
9. Given that $x_1 = 57$, $x_2 = 68$, and $x_3 = 32$, let $x_n = x_{n-1} - x_{n-2} + x_{n-3}$ for $n \geq 4$. Find x_{2013}.
10. Two squares are put side by side such that one vertex of the larger one coincides with a vertex of the smaller one. The smallest rectangle that contains both squares is drawn. If the area of the rectangle is 60 and the area of the smaller square is 24, what is the length of the diagonal of the rectangle?
11. On a field trip, 2 professors, 4 girls, and 4 boys are walking to the forest to gather data on butterflies. They must walk in a line with following restrictions: one adult must be the first person in the line and one adult must be the last person in the line, the boys must be in alphabetical order from front to back, and the girls must also be in alphabetical order from front to back. How many such possible lines are there, if each person has a distinct name?
12. Flatland is the rectangle with vertices A, B, C, and D, which are located at $(0, 0)$, $(0, 5)$, $(5, 5)$, and $(5, 0)$, respectively. The citizens put an exact map of Flatland on the rectangular region with vertices $(1, 2)$, $(1, 3)$, $(2, 3)$, and $(2, 2)$ in such a way so that the location of A on the map lies on the point $(1, 2)$ of Flatland, the location of B on the map lies on the point $(1, 3)$ of Flatland, the location of C on the map lies on the point $(2, 3)$ of Flatland, and the location of D on the map lies on the point $(2, 2)$ of Flatland. Which point on the coordinate plane is the same point on the map as where it actually is on Flatland?
13. S is a collection of integers such that any integer x that is present in S is present exactly x times. Given that all the integers from 1 through 22 inclusive are present in S and no others are, what is the average value of the elements in S?

14. In rectangle $PQRS$ with $PQ < QR$, the angle bisector of $\angle SPQ$ intersects $\overline{SQ}$ at point T and $\overline{QR}$ at U. If $PT : TU = 3 : 1$, what is the ratio of the area of triangle PTS to the area of rectangle $PQRS$?

15. For a function $f(x) = Ax^2 + Bx + C$, $f(A) = f(B)$ and $A + 6 = B$. Find all possible values of B.

16. Let α be the sum of the integers relatively prime to 98 and less than 98 and β be the sum of the integers not relatively prime to 98 and less than 98. What is the value of α/β?

17. What is the value of the series $\frac{1}{3} + \frac{3}{9} + \frac{6}{27} + \frac{10}{81} + \frac{15}{243} + \ldots$?

18. A bug starts at $(0, 0)$ and moves along lattice points restricted to (i, j), where $0 \le i, j \le 2$. Given that the bug moves 1 unit each second, how many different paths can the bug take such that it ends at $(2, 2)$ after 8 seconds?

19. Let $f(n)$ be the sum of the digits of n. How many different values of $n < 2013$ are there such that $f(f(f(n))) \neq f(f(n))$ and $f(f(f(n))) < 10$?

20. Let A and B be points such that $\overline{AB} = 14$ and let ω_1 and ω_2 be circles centered at A and B with radii 13 and 15, respectively. Let C be a point on ω_1 and D be a point on ω_2 such that $\overline{CD}$ is a common external tangent to ω_1 and ω_2. Let P be the intersection point of the two circles that is closer to $\overline{CD}$. If M is the midpoint of $\overline{CD}$, what is the length of segment $\overline{PM}$?